The Frege-Hilbert Controversy in Context

Abstract for the Twelfth French PhilMath Workshop

In a couple of letters Frege and Hilbert discussed the status of geometry and geometrical objects around 1900. After Hilbert terminated the conversation, Frege publishes a series of papers "Über die Grundlagen der Geometrie" in order to make his concerns about Hilberts attempt public.[1]

The controversy is widely conceived as one between a rather conservative 19th century mathematician – Frege – and a young man, who represents a new age of mathematics – Hilbert. Hilbert invented in his *Festschrift* a completely new understanding of axiomatization in mathematics. Frege had trouble with this new picture. Especially conservative seems his remark on non-euclidean geometry in an unpublished note, where he compare non-euclidean geometry to alchemy and astrology.[2]

For Frege geometrical axioms express truths, whose source of knowledge is pure intuition. Hilbert, on the other hand, seemingly detaches geometry completely from intuition. In a letter to Frege he wirtes: "If I think of my points as a system of some things, for example the system: love, law, chimney sweeper... , and then think of all of my axioms as relations among these things, then my sentences, i.e. that of Pythagoras hold of these things as well."[3]

In my talk I will reevaluate the relation between Hilbert and Frege. I will show, that both are deeply rooted in the geometry of 19th century, and that both even react to the same authors. In particular, I will argue that Freges position on non-euclidean geometry is widely misconceived. Hilbert on the other hand did not as radically break the connection between geometry and intuition in his *Festschrift* as some interpretations (among others that of Frege) suggest. Therefore Freges and Hilberts remark will be put in the context of 19th century debates among German mathematician.

I will draw the attention to the fact that non-Euclidean geometry was an intensively discussed topic in Göttingen, were Frege earned his PhD. I will show that Frege positively refers to non-Euclidean geometry in § 14 of *Grundlagen*,

[1] Originally Frege wanted to publish the letters of himself and Hilbert instead. (He suggested in his letter to Hilbert form January the 6th in 1900.) However, Hilbert did refute to do so. Thus, Frege decided to publish his concerns in a different way. (Über die Grundlagen der Geometrie, p. 319)

[2] "Über Eukldische Geometrie", in: Kleine Schriften p. 184, Translation in: Posthumous Writings p. 169

[3] "Wenn ich unter meinen Punkten irgendwelche Systeme von Dingen, z. B. das System: Liebe, Gesetz, Schornsteinfeger ..., denke und dann nur meine sämtlichen Axiome als Beziehungen zwischen den Dingen annehme, so gelten meine Sätze, z. B. der Satz vom Pythagoras auch von diesen Dingen." (Letter to Frege from December 29th 1899)

where he argues that geometry has a different epistemic status compared to geometry: Since non-euclidean geometry is non-contradictory, geometry rests on intuition. Since one cannot find an alternative arithmetic, on the other hand, arithmetic does not rely on intuition, but can be deduced from logic alone. I will show, that a quite similar line of argumentation can be found in the work of influential mathematicians such as Riemann and Gauß. Furthermore, we will see that even Hilbert takes up this line of argumentation in his early works on geometry.

In the second part of my talk, I will show how both, Frege and Hilbert, were influenced by the dispute between analytic and synthetic geometers. My focus will be on the topic of coordinization and extensional elements, which is mentioned in the early mathematical writing of Hilbert as well as Frege.

Freges early attempts to find euclidean representations for imaginary objects, show that the idea of finding different interpretations for the same expression, is not completely alien to Frege.

The topic of coordinization is addressed by Frege and Hilbert quite differently: Hilbert was interested in finding the axiomatic preconditions of the identification of points and numbers. He therefore proved the dependence on the archimedean axiom in the *Festschrift*.[4] Frege on the other hand, utilized the homogenous coordinate system of Julius Plücker extensively in his thesis, his lectures on analytic geometry he gave in Jena, and a few smaller mathematical writings. This kind of coordinate system tries to capture methods and results of synthetic geometry within the analytic approach. This attempt resembles Hilbert attitude insofar as it tries to overcome the strict distinction between analytic and synthetic geometry. It does, however, presuppose the possibility to identify points and numbers naively, by just introducing an external metric.

I will argue, that in this different way to attempt the topic of coordinization, is symptomatic for Freges and Hilberts different view on the relation between geometry and intuition: For Hilbert intuition is analyzed by axiomatic geometry. For Frege, however, intuition is the source of geometry. A coordinate system which is convenient to express relationships between the geometrical objects, which we are familiar with by intuition, thus does not need another kind of verification.

This diagnosis makes it then possible to solve the tension between Freges odd remarks in his late note on non-euclidean geometry note and his positive attitude to euclidean geometry in other writings: In his "Notiz über euklidische Geometrie" Frege primarily struggles with the modern understanding of axiom, which can also be found in Hilberts work. For Frege, axioms express truths about geometrical objects, given to us in intuition. If the parallel axiom is true, than there can be no axiom negating it, because it would be false, but there can, by definition, be no false axioms. Frege is, however, able to distinguish between making a judgment and just uttering a sentence. His *Begriffsschrift* even provides a sign to express this difference. Finding euclidean interpretations

[4]This was investigated in depth in Giovanni, Eduard. *Bridging the gap between analytic and synthetic geometry: Hilbert's axiomatic approach.* (2016)

for non-euclidean expressions (like in his thesis) and pointing to the fact, that the sentences of non-euclidean geometry do not contradict the basic laws of logic, is thus in accordance with his negative response to non euclidean axioms.

The same holds for Freges remarks on Hilberts axiomatics in general, as it is expressed in "Über die Grundlagen der Geometrie". The idea of different interpretation of the same sentence is not alien to Frege. But, firstly, this does not hold for axioms, because they express truths. And, secondly, there are only euclidean objects, which can serve as interpretations for any kind of geometrical sentences. Whereas for Hilbert geometrical objects do not serve as an interpretation. In the contrary, any model for geometrical axioms, should behave just *as* geometrical objects. And in this way our intuition of these objects is "analyzed".

The result of my talk cant thus be summarized as follows:

(1.) Frege and Hilbert are not mathematicians of completly different ages. They do share common heritage in 19th century projective geometry: Frege as well as young Hilbert took over the view on the different epistemic status of geometry and arithmetic from Gauß and Riemann. Both thought about the topic of coordinization and extensional objects and were thereby inspired by earlier 19th century geometers.

(2.) Frege is neither unable to understand that non-euclidean geometry is non contradictory. He simply thinks it is false. Nor is Frege unable to understand, how one could attribute different meanings to a sign. He just thinks that an axiom must be true and for attribute truth to a sentence its concept names must have meaning.

(3.) Hilbert does not detach Geometry from intuition. But axioms do not express true sentences about objects given to us in intuition. Rather intuition is captured by axioms.