Bolzano's Mathematical Infinite

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Bernard Bolzano's Paradoxien des Unendlichen (PU for short) is one of his most widely read works. Even so, it is still a fascinatingly fresh read for the contemporary philosopher of maths, as its standing with respect to the broader history of mathematical infinity remains little understood (cf., e.g., Berg 1962, 1992; Šebestík 1992; Rusnock 2000). This paper gives a new interpretation of Bolzano's "calculation of the infinite" as presented in PU, §§ 29-33. We claim that, contrary to a widespread view (e.g. Berg 1962) Bolzano's treatment of the size of infinite (countable) collections is not a defective (Berg) or inconsistent (Šebestík) anticipation of Cantorian set theory, but rather a sketch of an alternative theory of the infinite. The key to understanding PU §§29-33, we argue, is realizing that Bolzano is comparing the size of infinite sums, not infinite sets or set-like collections. After carefully analyzing the relevant passages in the PU, we bolster our interpretation by modelling Bolzano's computations with infinite sums with an ultrapower construction. While ultrapowers have recently become central tools in developing non-Cantorian accounts of the size of infinite sets more generally (Benci and Di Nasso 2003; Trlifajová 2018), our specific construction allows us to reproduce Bolzano's more surprising results while remaining faithful to his own reasoning. This builds a strong case that Bolzano's views on the countable infinite can be read as being consistent yet different from Cantor's in a deep way.