Heuristic rigor / constraints / control

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What is a theorem?

A theorem is:

- a statement
- a proof

We may qualify the statement with adjectives: it is true, beautiful, interesting, useful ...

We may qualify the proof with adjectives: it is correct, instructive, simplified, technical ...
More mathematical results

We may also wish to prove a *conditional theorem*, i.e. a theorem which depends on a statement deemed to be correct or discussed because it presents a major issue (example: the Riemann hypothesis).

We may wish to produce a different (simpler, more conceptual) proof of an already existing theorem.

The reasons why a conditional theorem is accepted (and published) as an interesting mathematical result can be that its statement is valuable, or because it says something on the important conjecture it relies on.

In other words, the conditional theorem can be seen as an implication: big conjecture $\Rightarrow$ conditional statement. This is a potential way to disprove the conjecture.
We are guided

Why do we want to prove a theorem?

- Because there is a mathematical/scientific issue
  and
- Because we think we "know" how to do it!

The criteria which make it possible to evaluate a statement or theorem are of various natures: esthetics, utility, and this from a point of view internal or external to mathematics.

There are also arguments of authority or "social" recognition.
Some examples

- Fermat’s last theorem: it is at the same an elementary statement and an extremely hard proof; the proof itself had important further applications (more technical).

- The fundamental lemma in automorphic forms is a crucial result in the field, whose statement is hardly understood by non-specialist mathematicians.

- Stability of the hydrogen atom: it is an important theorem from mathematical analysis (spectral theory), which uses a physical model, but it is not clear whether physicists care.
We are framed

We are framed by various factors:

- by anticipating technical difficulties,
- by logical implications (we make multiple arguments by the absurd),
- by our previous experiences

What is the issue? Mathematical creativity.

Where does rigor play a role?

In all cases, the rigor in question at this level is rigor of evaluation, in connection with mathematical creativity. It is not in opposition to the latter, it is rather an economy of intellectual means.
Writing mathematics to share them

This part is about rigor in producing and writing of published mathematics.

We will focus here on the constraints related to the rules that we allow ourselves (or not) and the constraints related to the object of study itself.
The game’s rules: implicit choices

A first form of rigor in mathematical production can be summed up by sticking to the chosen rules.

Indeed, even if it is often implicit, we have the possibility of specifying the corpus of axioms that we do not allow ourselves.

If we do not, logicians can remind us of these ”choices”!
The game’s rules: example of set theory

Set theory is the canonical example. It is well known that there are different models.

A small paradox (or provocation?): Bourbaki’s choices in this area. It is a very powerful set theory (use of Hilbert’s symbol), but which bypasses category theory (which has recently found its limits, especially for the first books on algebraic topology).

Please note, these two choices did not take place at the same time.
The object of study: classifications

Choosing an object of study involves choosing a level of generality and sticking to it.

This is a question that usually arises in classification problems. To speak a little technically, when we want to classify we must first clearly identify the objects that interest us, then adjust the equivalence relation used.

Examples:

1. Classifications of surfaces, more generally classifications in algebraic topology.
2. Classifications of groups in algebra and in geometry.
The object of study: subsequent changes and generalization

Sticking to the object of study, in the sense of pushing the work without deviating from the initial definition, is a standard aspect of mathematical activity.

This is quite crucial when doing modeling (both with techniques from fundamental mathematics and from applied mathematics). The object of study is often complex, or far from mathematics, and it is necessary to record the simplifications made. This allows for later generalizations, even unsuspected ones.

Examples:
1. Heat equation (Fourier series, Poincaré’s conjecture).
2. Standard model of elementary particles (based on symmetry, group-theoretic, considerations).

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The object of study: two further examples

Among the examples of questionable choice of objects of study, or rather of poor adjustment of the level of generality, we can cite the rather famous example of Bourbaki’s mistake in integration theory. This theory cannot account for modern theory of probability.

Another question, which may be pointed out to philosophers, is that of the corpus of mathematical notions studied by Cavaillès and Lautman, respectively.
Validation

One of the meanings of a posteriori rigor is that of the validation of scientific results.

This is undoubtedly the most widespread and the most normative meaning when we talk about mathematical rigor.

How does it work?

- Peer proofreading: an anonymous referee will find all mistakes and will make the submitted publication more readable.
- In the case of important results: an institutional distinction (but think of Poincaré’s mistake on the three body problem).
- Repeated and various uses of the result in question.
The case of modeling

The interface with other disciplines makes validation questions stronger. In fact, questions of validation of the model as a mathematical object and as an adequate tool for the other scientific discipline mobilized are played out at the same time.

A small paradox: the term ”control theory” in applied mathematics.
In the control part, we can obviously think of computer-aided mathematics. This point was originally included at the end of the first part (framing intuitions), but after all it was more a matter of verification (checking intuitions).

It was ultimately not discussed, because the use of computer itself is evolving rapidly at the moment: there is of course proof control by computers, but also the beginnings of automatic proof are being experimented by now.
This review of the terms "rigor" in mathematics may suffer from a bias, namely from the fact that the entry into the subject deals with theorems and is not about theories.

One can imagine that a change of scale would enrich the question, but one can already notice that the term "theory" comes up several times in the part about constraints.