Bolzano's Mathematical Infinite

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Overview

- Bolzano's calculation of the infinite (§§29 – 33 of his *Paradoxes of the Infinite*) is widely seen as a fascinating yet ultimately unsuccessful anticipation of Cantorian transfinite arithmetic. Unlike Cantor, Bolzano appeals to part-whole intuitions in determining order relationships between infinite quantities. Bolzano's calculation of the infinite is then usually interpreted as a fundamentally flawed attempt to apply part-whole intuitions in order to determine size relationships between sets.

- We aim to challenge this narrative by offering a reinterpretation of Bolzano's text as a theory of infinite *sums*, rather than a theory of infinite sets. We argue that Bolzano's appeal to part-whole reasoning in the context of infinite sums is both coherent and fruitful, and allows him to sketch an original theory of the infinite.

- Using modern model-theoretic techniques, we provide a formalization of Bolzano's theory that establishes the consistency of his views and highlights the coherence of his arguments.

Plan

1. Background
2. Bolzano's Calculation of the Infinite
   - Infinite Sums
   - Grandi's Series
   - The Sum of all Squares
3. Formalizing the Bolzanian Infinite
   - An Ultrapower construction
   - Bolzanian products

Background

- Bolzano's *Paradoxes of the Infinite* (1848) is a posthumous booklet in which the author attempts to solve various mathematical and philosophical problems surrounding the infinite.

- The *Paradoxes* are one of Bolzano's most famous writings, in particular because of the "Calculation of the Infinite" presented in §§29 – 33.

- These paragraphs are often read as an imperfect anticipation of Cantor's transfinite arithmetic by most Bolzano scholars (e.g., Berg, Sebestik), as well as by Cantor himself.

- According to this view, Bolzano should be credited for trying to develop an arithmetic of actually infinite collections. But he only obtains partial or even inconsistent results, because he fails to arrive at the modern notion of cardinality.

Bolzano is perhaps the only one who confers a certain status to actually infinite numbers, or at least they are often mentioned [by him]; nevertheless I completely and wholly disagree with the way in which he handles them, not being able to formulate a proper definition thereof, and I consider for instance §§29-33 of that book as untenable and wrong. For a genuine definition of actually infinite numbers, the author is lacking both the general concept of power, and the accurate concept of number. It is true that the seeds of both notions appear in a few places in the form of special cases, but it seems to me he does not work his way through to full clarity and distinction, and this explains several contradictions and even a few mistakes of this worthwhile script. (Cantor, "Über unendliche, lineare Punktmannichfaltigkeiten – 5. " (1883))

- Following Cantor, most commentators read Bolzano's calculation of the infinite as an essentially misguided attempt to develop an arithmetic of infinite sets based on *part-whole reasoning* (the whole is always greater than its proper parts) rather than on the *bijection principle* (two collections have the same size if and only if they are bijectable with one another).

- At least since Galileo, the two intuitions are known to yield inconsistent results in the case of infinite collections: the set of all square numbers, for example, is both a proper subset of, and bijectable with, the set of all natural numbers.
- Bolzano discusses variations of Galileo's paradox in $\S 20-24$ of the *Paradoxes*. He endorses part-whole reasoning, and argues that the existence of a one-to-one correspondence between two infinite collections is not enough to conclude the equality of their sizes.
- As long as Cantor's transfinite arithmetic is viewed as the only viable theory of the infinitely large, against which all previous attempts to deal with infinite quantities must be compared, Bolzano's calculation of the infinite can only be a mathematical and conceptual dead end.
- Our goal is to offer a reappraisal of Bolzano's theory, that highlights its specificity and its conceptual independence from Cantor's transfinite arithmetic.

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Overview

- Bolzano's calculation of the infinite ($\S\S 29 - 33$) discusses both infinitely large and infinitely small quantities.

- Since we're interested in his views on the infinitely large, we will focus on $\S\S 29, 32$ and 33. We wish to establish the following three claims:

1. Bolzano is developing a theory of infinite *sums*, not of infinite *sets*. His sums have two distinct quantitative aspects: a value and a number of terms.

2. This distinction plays a fundamental role in addressing old and new paradoxes about divergent infinite sums.

3. Bolzano's computations are guided by a form of part-whole reasoning about infinite sums, which differs from, but is nonetheless consistent with part-whole reasoning about sets.

- In §29, Bolzano introduces several infinite sums and discusses their relationships. Starting from the sequence (*Reihe*) of natural numbers, he distinguishes the **sum** of all natural numbers:

$$\overset{1}{S} = 1 + 2 + 3 + \dots \ \textit{in inf.}$$

and the **number** (*Menge*) of all natural numbers:

$$\overset{0}{N} = 1^0 + 2^0 + 3^0 + \dots \ \textit{in inf.},$$

obtained from $\overset{1}{S}$ by uniformly raising each term to the 0^{th} power, thus viewing them as units.

- Bolzano claims that $\overset{1}{S}$ is "far greater" than $\overset{0}{N}$. Two infinite sums, however, may also designate infinite quantities that have a finite difference. Bolzano considers the number of all natural numbers greater than some fixed n:

$$\overset{n}{N} = (n+1)^0 + (n+2)^0 + \dots \ \textit{in inf.},$$

and claims that:

$$\overset{0}{N} - \overset{n}{N} = 1^0 + 2^0 + \dots + n^0 = n.$$

- Bolzanian sums therefore have two associated quantitative aspects: their *value*, which Bolzano calls “the quantity they designate”, and their number of terms (*Gliedermenge*).

- Given an infinite sum $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \ldots$ *in inf.*, the procedure to determine the *Gliedermenge* of α is the following:
  1. Raise all terms in α to the 0^{th} power, thus viewing them as units;
  2. Compute the value of the resulting infinite sum.

- Infinite sums with a distinct *Gliedermenge* and value can be obtained from others.

- Infinite sums can not only be added, but also multiplied, leading to infinities *of higher order*, like $(\overset{0}{N})^2, (\overset{0}{N})^3, \ldots$

- Bolzano doesn't discuss products of infinite sums much, but there are reasons to think this operation is not commutative. He also explicitly reject an infinitary version of Gauss's summation theorem, namely that:

$$2\overset{1}{S} = \overset{0}{N}.(\overset{0}{N} + 1).$$

Grandi's Series

- In §32, Bolzano discusses certain pathological geometric series known as Grandi's series.

- A *geometric* series is an infinite sum in which the ratio between consecutive terms is constant, i.e., a series of the form $\sum_{n=0}^{\infty} ar^n$ for some real numbers a and r.

- If $|r| < 1$, the series converges to $\frac{a}{1-r}$. Bolzano considers the case $r = -1$, i.e., the infinite sum:

$$G_a := a - a + a - a + \dots \ \textit{in inf.}$$

- The case $a = 1$ was introduced by Grandi in 1703, who gave a geometric argument for $G_1 = \frac{1}{2}$. In general, by analogy with geometric series for which $|r| < 1$, one might be tempted to conclude that:

$$a - a + a - a + \dots = \frac{a}{1-(-1)} = \frac{a}{2}.$$

- Bolzano, however, objects to an algebraic proof of this result published in 1830 in Gergonne's *Annales*.

- The proof goes as follows:

$$\begin{aligned} G_a &= a - a + a - a + \ldots \textit{ in inf.} \\ &= a - (a - a + a - \ldots \textit{ in inf.}) \\ &= a - G_a \\ G_a &= \frac{a}{2}. \end{aligned}$$

- Bolzano gives two reasons to reject this proof. First, G_a is not equal to the infinite sum within parentheses, because the two sums do not have the same *Gliedermenge*. Instead, the sum within parentheses should be equal to $-(G_a - a)$.

- Second, G_a does not even designate an actual quantity, because changing the order in which the terms of G_a are summed also changes the value it should have:

$$\begin{aligned} (a - a) + (a - a) + \ldots \textit{ in inf.} &= 0 + 0 + \ldots \textit{ in inf.} &&= 0; \\ a + (-a + a) + (-a + a) + \ldots \textit{ in inf.} &= a + 0 + 0 + \ldots \textit{ in inf.} &&= a. \end{aligned}$$

- The *Gliedermenge* / value distinction is therefore instrumental in solving an old problem about infinite sums.

The Sum of all Squares

- In §33, Bolzano compares the sum of all natural numbers

$$\overset{1}{S} := 1 + 2 + 3 + \ldots \textit{ in inf.}$$

  with the sum of all squares

$$\overset{2}{S} := 1^2 + 2^2 + 3^2 + \ldots \textit{ in inf.} = 1 + 4 + 9 + \ldots \textit{ in inf.}$$

- Even though all terms in $\overset{2}{S}$ appear in $\overset{1}{S}$, and not vice-versa, Bolzano arrives at the surprising conclusion that $\overset{2}{S}$ is infinitely greater than $\overset{1}{S}$.
- Bolzano gives a detailed argument, in which he seems to be working out a sufficient criterion for determining part-whole relationships between infinite sums:

1 Bolzano argues first that, since $\overset{2}{S}$ is obtained from $\overset{1}{S}$ by raising each term to the power of 2, both sums have the same *Gliedermenge*.

2 Next, $\overset{1}{S}$ can be seen to be a *proper part* of $\overset{2}{S}$, by observing that all but the first term in the following sum are positive:

$$\overset{2}{S} - \overset{1}{S} = (1-1) + (4-2) + (9-3) + \dots \textit{ in inf.},$$

3 More generally, for any natural number n, the sum:

$$\overset{2}{S} - n\overset{1}{S} = (1-n) + (4-2n) + (9-3n) + \dots \textit{ in inf.}$$

contains only finitely many non-positive terms, and is therefore positive.

4 Bolzano concludes that any finite multiple of $\overset{1}{S}$ is a proper part of $\overset{2}{S}$, and hence that $\overset{2}{S}$ is infinitely greater than $\overset{1}{S}$.

- Taking stock: given two infinite sums α and β of same *Gliedermenge*, β is greater than α iff α is a *proper part* of β, i.e., iff their difference $\beta - \alpha$ is positive.

- Bolzano must therefore answer two questions in order to determine part-whole relationships between two infinite sums α and β:
  1. What their difference $\beta - \alpha$ is;
  2. When a given infinite sum γ positive.

- He gives a (partial) answer in §33:
  1. Given two infinite sums $\alpha := \alpha_1 + \alpha_2 + \alpha_3 + \dots$ *in inf.* and $\beta := \beta_1 + \beta_2 + \beta_3$, their difference is computed termwise:

     $$\beta - \alpha := (\beta_1 - \alpha_1) + (\beta_2 - \alpha_2) + (\beta_3 - \alpha_3) + \dots \textit{ in inf.}$$

  2. If all but finitely many terms in an infinite sum γ are positive, then γ is positive.

- We can therefore extract from this passage a part-whole principle for *infinite sums*:

  $(PW)_{sum}$ For any two infinite sums α, β, if all but finitely many terms in $\beta - \alpha$ are positive, then $\alpha < \beta$.

- It is worth contrasting this principle with one that could be obtained for *sets* along more familiar lines:
  1. Given two sets A and B, the difference $B \setminus A$ is the set of all elements in B that are not in A;
  2. A set A has *positive size* iff A is non-empty.
  3. Therefore, a set A is a *proper part* of another set B if $B \setminus A$ has positive size, i.e., if $A \subsetneq B$. Hence:

     $(PW)_{set}$ For any two sets A and B, if $A \subsetneq B$, then $size(A) < size(B)$.

- Unlike **$(PW)_{sum}$**, **$(PW)_{set}$** cannot be applied to the quantities designated by $\overset{1}{S}$ and $\overset{2}{S}$, and would contradict Bolzano's result if applied to the *Gliedermengen* of both sums.

- Bolzano's criterion for comparing two infinite sums requires both sums to have the same *Gliedermenge*. In particular, $\overset{1}{S}$ and $\overset{2}{S}$ have the same *Gliedermenge*, as the terms in the latter are obtained by raising the terms in the former to the power of 2.

- But if $\overset{1}{S}$ is the sum of all natural numbers, and $\overset{2}{S}$ the sum of all squares, and both have the same number of terms, doesn't this imply that there are as many squares as there are natural numbers, thus contradicting part-whole?

- We think that this conclusion can be avoided by analogy with Bolzano's computation of $\overset{n}{N}$ in §29.

- Recall that Bolzano claims that

$$\overset{0}{N} - \overset{n}{N} = 1^0 + 2^0 + \ldots + n^0 = n.$$

- Given **(PW)$_{\textbf{sum}}$**, this suggests that the number of all numbers greater than n is accurately determined by *erasing* the first n terms from $\overset{0}{N}$:

$$\overset{n}{N} = \underbrace{\quad + \ldots + \quad}_{n \text{ times}} + (n+1)^0 + (n+2)^0 + \ldots \textit{ in inf.}$$

- Unlike uniformly raising all terms to a given power, this operation *does* change the *Gliedermenge* of a sum.

- Similarly, the number of squares should not be computed as the value of the sum:

$$1^0 + 4^0 + 9^0 + \ldots \textit{ in inf.},$$

obtained by raising all terms in $\overset{2}{S}$ to the 0^{th} power, but rather, as the value of the sum:

$$1^0 + \quad + \quad + 4^0 + \quad + \ldots \textit{ in inf.},$$

obtained by erasing all non-squares from $\overset{0}{N}$.

Summing up

- Bolzano's calculation of the infinite should be interpreted as a theory of *infinite sums*, rather than as a proto-transfinite arithmetic of infinite sets.

- Bolzanian infinite sums have two distinct quantitative aspects: their *Gliedermenge*, and their value, i.e., the quantity they designate. Neither notion maps onto the modern notion of cardinality.

- Bolzano addresses old and new problems about infinite sums by introducing operations on infinite sums that may impact either of their associated quantities:
    1. Uniformly raising all terms in an infinite sum to a certain power changes its value, but not its *Gliedermenge*;
    2. Permuting the order of its summands may change its *Gliedermenge*, but not its value;
    3. Erasing certain terms in a sum changes both its *Gliedermenge* and its value.

- Bolzano's computations obey a form of part-whole principle for infinite sums, but are nonetheless consistent with part-whole reasoning for sets.

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An Ultrapower construction

- Our proposal: to model Bolzano's computations using an ultrapower $\mathbb{Z}_{\mathscr{U}}$ of the integers $(\mathbb{Z}, +, -, <, 0)$, using a non-principal ultrafilter $\mathscr{U}$ on the set of positive integers ω^+.

- Intuitively, we may think of $\mathscr{U}$ as a collection of properties describing a new natural number "at infinity", an ideal vantage point from which we can observe infinite sums as completed.

- Elements in $\mathbb{Z}_{\mathscr{U}}$ are (equivalence classes of) functions from ω^+ into $\mathbb{Z}$. Operations are defined pointwise, and for any $\alpha, \beta : \omega^+ \to \mathbb{Z}$, $\alpha^* < \beta^*$ iff $\{i \in \omega^+; \alpha(i) < \beta(i)\} \in \mathscr{U}$.
  By Łoś's Theorem, $\mathbb{Z}_{\mathscr{U}}$ is a linearly ordered additive group.

- To any Bolzanian infinite sum $\alpha := \alpha_1 + \alpha_2 + \alpha_3 + \dots$ *in inf.*, we associate an *approximating sequence* $\sigma(\alpha) : \omega^+ \to \mathbb{Z}$ defined by

$$\sigma(\alpha)(i) = \sum_{k=1}^{k=i} \alpha_k.$$

- For sequences with a distinct *Gliedermenge*, we "fill in the gaps" with 0*s* so that all infinite sums may be compared.

- Finally, we interpret the quantity designated by α as the equivalence class α of $\sigma(\alpha)$ in $\mathbb{Z}_{\mathcal{U}}$.

Bolzanian Infinite Sum	Sequence Representation	Approximating Sequence	Infinite Quantity
$1^0 + 2^0 + \dots$ *in inf.*	$\overset{0}{N} = (1, 1, 1, 1, \dots)$	$\sigma(\overset{0}{N}) = (1, 2, 3, 4, \dots)$	$\overset{0}{N} = \sigma(\overset{0}{N})^*$
$(n+1)^0 + (n+2)^0 + \dots$ *in inf.*	$\overset{n}{N} = (\underbrace{0, \dots, 0}_{n \text{ times}}, 1, 1, \dots)$	$\sigma(\overset{n}{N}) = (\underbrace{0, \dots, 0}_{n \text{ times}}, 1, 2, 3, \dots)$	$\overset{n}{N} = \sigma(\overset{n}{N})^*$
$1 + 2 + 3 + \dots$ *in inf.*	$\overset{1}{S} = (1, 2, 3, 4, \dots)$	$\sigma(\overset{1}{S}) = (1, 3, 6, 10, \dots)$	$\overset{1}{S} = \sigma(\overset{1}{S})^*$
$1^n + 2^n + 3^n \dots$ *in inf.*	$\overset{n}{S} = (1^n, 2^n, 3^n, 4^n, \dots)$	$\sigma(\overset{n}{S}) = (1^n, (1^n + 2^n), \dots)$	$\overset{n}{S} = \sigma(\overset{n}{S})^*$
$a - a + a - a + \dots$ *in inf.*	$G_a = (a, -a, a, -a, \dots)$	$\sigma(G_a) = (a, 0, a, 0, \dots)$	$G_a = \sigma(G_a)^*$

Table 1: Representation of Bolzanian sums in $\mathbb{Z}_{\mathcal{U}}$

Some Results

1. Infinite quantities can have a finite difference. For any natural number n, $\mathbb{Z}_{\mathscr{U}} \models \overset{0}{N} - \overset{n}{N} = n$.

2. Some infinite quantities are infinitely greater than some others. For any natural number i, $\mathbb{Z}_{\mathscr{U}} \models i\overset{0}{N} < \overset{1}{S}$ and for any natural numbers i, n, $\mathbb{Z}_{\mathscr{U}} \models i\overset{n}{S} < \overset{n+1}{S}$.

3. Grandi's series are indeterminate: for any integer a, $\mathbb{Z}_{\mathscr{U}} \models G_a = 0$ if $Evens \in \mathscr{U}$, and $\mathbb{Z}_{\mathscr{U}} \models G_a = a$ if $Odds \in \mathscr{U}$.

4. $\mathbf{(PW)_{sum}}$ holds: for any two Bolzanian sums α, β: $\mathbb{Z}_{\mathscr{U}} \models \alpha < \beta$ if $\{i \in \omega^+ : \sigma(\beta - \alpha)(i) > 0\}$ is cofinite.

5. A version of $\mathbf{(PW)_{set}}$ also holds. Assuming that the size of a set $A \subseteq \omega^+$ is interpreted as the value χ_A, we have that for any $A, B \subseteq \omega^+$, $A \subset B \Rightarrow \mathbb{Z}_{\mathscr{U}} \models \chi_A < \chi_B$.

These results are immediate consequences of the componentwise definition of addition and subtraction of Bolzanian sums, and of reasoning modulo a non-principal ultrafilter.

- In §29, Bolzano mentions that infinite quantities can also be multiplied. His examples are

$$(\overset{0}{N})^2 = \overset{0}{N} + \overset{0}{N} + \overset{0}{N} + \dots \textit{ in inf.}$$

$$(\overset{0}{N})^3 = (\overset{0}{N})^2 + (\overset{0}{N})^2 + (\overset{0}{N})^2 \dots \textit{ in inf.}$$

$$\dots$$

- In our model, we could attempt to define the product of two Bolzanian sums α and β componentwise, i.e. by defining $\alpha \times \beta$ as (the equivalence class of) the function $i \mapsto \alpha(i) \times \beta(i)$.
- This would make $\mathbb{Z}_{\mathscr{U}}$ a commutative ordered ring, and yield a structure very similar to recent proposals (Benci and di Nasso) in the literature on non-Cantorian theories of infinite collections.
- But there are two major shortcomings to this idea.

1 This definition conflicts with **(PW)$_{\text{sum}}$**. Indeed, if we let

$$\alpha := 1 + 3 + 5 + 7 + \dots \textit{ in inf.},$$

then $\mathbb{Z}_{\mathscr{U}} \models \alpha = \overset{0}{N} \times \overset{0}{N}$. On the other hand, all terms in

$$(\overset{0}{N})^2 - \alpha = (\overset{0}{N} - 1) + (\overset{0}{N} - 3) + (\overset{0}{N} - 5) + \dots \textit{ in inf.}$$

are positive, so from **(PW)$_{\text{sum}}$** we should conclude that $\alpha < (\overset{0}{N})^2$.

2 As a direct consequence of Gauss's summation theorem, we have that $\mathbb{Z}_{\mathscr{U}} \models 2\overset{1}{S} = \overset{0}{N} \times (\overset{0}{N} + 1)$. But Bolzano explicitly argues in §29 that the summation theorem does not hold in the infinite case.

- Our solution is take Bolzano at face value when he writes that

$$(\overset{0}{N})^2 = \overset{0}{N} + \overset{0}{N} + \overset{0}{N} + \ldots \textit{ in inf.}$$

- More precisely, $(\overset{0}{N})^2$ is obtained by summing $\overset{0}{N}$-many times the infinite quantity $\overset{0}{N}$. It is an infinite sum of infinite quantities.

- Formally, this means that we should represent $(\overset{0}{N})^2$ by (the equivalence class of) the function $i \mapsto \underbrace{\overset{0}{N} + \ldots + \overset{0}{N}}_{i \text{ times}}$. More generally, we must work inside an ultrapower $(\mathbb{Z}_{\mathscr{U}})^2$ of $\mathbb{Z}_{\mathscr{U}}$, and define $\alpha.\beta$ as (the equivalence class of) the function $i \mapsto \underbrace{\beta + \ldots + \beta}_{\sigma(\alpha)(i) \text{ times}}$.

Some consequences of this definition:

1. Bolzanian products are not commutative. We view this as motivated by **(PW)$_{\text{sum}}$**. Compare:

$$\overset{0}{N}.\overset{1}{S} = \overset{1}{S} + \overset{1}{S} + \overset{1}{S} + \ldots \textit{ in inf.}$$

$$\overset{1}{S}.\overset{0}{N} = \overset{0}{N} + 2\overset{0}{N} + 3\overset{0}{N} + \ldots \textit{ in inf.}$$

As all terms in $\overset{0}{N}.\overset{1}{S} - \overset{1}{S}.\overset{0}{N} = (\overset{1}{S} - \overset{0}{N}) + (\overset{1}{S} - 2\overset{0}{N}) + (\overset{1}{S} - 3\overset{0}{N})$ are positive, by **(PW)$_{\text{sum}}$** we should conclude that $\overset{1}{S}.\overset{0}{N} < \overset{0}{N}.\overset{1}{S}$.

2. The infinite summation theorem $(2\overset{1}{S} = \overset{0}{N}.(\overset{0}{N} + 1))$ does not hold. In fact, the product of any two positive infinite quantities α and β in $\mathbb{Z}_{\mathscr{U}}$ is always infinitely greater than any quantity in $\mathbb{Z}_{\mathscr{U}}$.

3. This definition makes explicit the idea that some infinite quantities are of genuine *higher-order* than others, in the sense that they can only be obtained by applying an infinitary operation to a collection of infinite quantities.

- In order to be able to define the product of any two Bolzanian quantities, we must iterate this construction countably many times, and take a direct limit $\mathbb{B}$ of this chain of ultrapowers.

$$\mathbb{Z} \longrightarrow \mathbb{Z}_{\mathscr{U}} \longrightarrow (\mathbb{Z}_{\mathscr{U}})^2 \longrightarrow (\mathbb{Z}_{\mathscr{U}})^3 \longrightarrow \ldots \mathbb{B}$$

- Since functions from ω^+ to $(\mathbb{Z}_{\mathscr{U}})^n$ can equivalently be described as functions from $(\omega^+)^{n+1}$ to $\mathbb{Z}$, we may define the product $\alpha.\beta : (\omega^+)^{n+m} \to \mathbb{Z}$ of two infinite quantities $\alpha : (\omega^+)^n \to \mathbb{Z}$ and $\beta : (\omega^+)^m \to \mathbb{Z}$ as the map $\overline{ij} \mapsto \alpha(\bar{i}) \times \beta(\bar{j})$ for any n-tuple $\bar{i}$ and any m-tuple $\bar{j}$.

- We obtain a well-behaved yet original structure: $(\mathbb{B}, +, -, ., <, 0, 1)$ is a non-commutative ordered ring that satisfies **(PW)$_{\text{sum}}$**.

Conclusion

- We offered a new interpretation of Bolzano's calculation of the infinite as a theory of infinite sums.
- Reintroducing some distance between Bolzano's project and Cantor's allows to cast the former into a much more favorable light.
- As we have argued, Bolzano's theory of infinite sums is coherent, fruitfully based on part-whole reasoning, and amenable to an original and straightfoward formal reconstruction.