Informal and absolute provability: from Kreisel and Gödel to Prawitz and Girard

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Twelfth French Philosophy of Mathematics Workshop 12 November 2020

(Catta LIRMM, Piccolomini CGGG) Informal and absolute provability

FPMW12 5/11/2020

- **Aim**: evaluating Kreisel's informal rigour and Gödel's absolute provability with respect to Prawitz's semantics of valid arguments and grounds, and Girard's proof-nets and Ludics.
- **Claims**: Prawitz's semantics share some points with Kreisel's informal rigour. Gödel's absolute provability reminds of Girard's approach. An indirect link between Prawitz and Gödel through a link between Prawitz and Girard.

Kreisel 1960, Ordinal logics and the characterization of informal concepts of proof \implies expansion of Turing's and Feferman's work on extensions of finitary recursion theories, towards a theory of the totality of such extensions and a concept of informal constructive proof. The principle of extension of recursive arithmetic is informal - as non-finitary - constructive.

Kreisel 1967, Informal rigour and completeness proof \implies Turing's work used for an *ex negativo* analysis: a paradigm of formal rigour. But

formal rigour does not apply to the discovery or choice of formal rules nor of notions. (Kreisel 1967)

Insofar as we move within a given formalism, we cannot "look beyond" its boarders, and resort to tools other than those available in it. The activity of detecting conceptual and deductive means that widen our system is **informal rigour**:

by analysing intuitive notions and putting down their properties [...] the intuitive notions are significant, be it in the external world or in thought. (Kreisel 1967)

Kreisel 1987, Church's thesis and the ideal of informal rigour \implies formal way of establishing definitions and properties = deduction from established knowledge; informal way = "inspection by the mind's eye".

Kreisel's examples: higher-order axiomatizations and independence results in set-theory, logical consequence and completeness, Markov's principles and Brouwer choice-sequences, non-standard models.

The formal and the informal are "sides of the same coin". We have a "dialectic" between formal and informal. Informal rigour overcome formalisation and the limiting results it suffers from, and approximates to more and more powerful analysis. We here principally refer to Gödel 1946, *Remarks before the Princeton bicentennial conference on problems in mathematics*, and rely upon Crocco's 2019 *Informal and absolute proofs: some remarks from a Gödelian perspective*.

A *weak* and a *strict* degree. Weak absoluteness is like Kreisel's informal rigour. Strict absoluteness is independent of language and domains of things.

With reference to proofs, Gödel envisages a hierarchy of systems obtained by transfinite iteration of certain operations, and then considers the hierarchy as a whole:

there cannot exist any formalism which could embrace all these steps; but this does not exclude that all the steps [...] could be described and collected together in some non-constructive way. (Gödel 1946)

But later on - although now wrt absolute definability - Gödel remarks that

the concepts arrived at [...] were not absolute in the strictest sense, but only wrt a certain system of things. (Gödel 1946) Gödel has a high opinion of Turing's work on computability. Crocco writes:

"absolute" means "non-relative to any particular formal system or formalized language", but there is also more than that. [...] Gödel considered Turing's definition "a miracle" [...] being independent of any language and formal system, it cannot be subject to any diagonalisation. It is also a miracle because it is strictly independent of any domain of things. (Crocco 2019)

Weak absolute proofs \implies transcends formal systems, no diagonalisation. Strict absolute proofs \implies no domain of things, no language at all. <u>Argument</u>: pair $\langle \Delta, \mathfrak{J} \rangle$ for Δ tree of formulas, nodes arbitrary inferences, \mathfrak{J} set of reduction procedures.

<u>Closed</u>: no free assumptions or variables. Open: otherwise.

Non-canonical steps justified by elements of $\mathfrak{J},$ for i,j = 1,2 and $i\neq j,$

$$\frac{\begin{array}{cc}\Delta_{1}\\\hline A_{i}\\\hline A_{1}\vee A_{2}\end{array}}{A_{i}} \begin{pmatrix} \vee_{I}^{i} \end{pmatrix} & \begin{array}{c}\Delta_{2}\\\hline A_{j} \rightarrow \bot\\\hline A_{i}\end{array} \longrightarrow \begin{array}{c}\Delta_{1}\\\hline A_{i}\end{array}$$

<u>Closed grounds</u>: induction on the complexity of the logical form, with primitive.

E.g. $f(x^A)$ transforms grounds for A into grounds for $B \Rightarrow \rightarrow Ix^A(f(x^A))$ ground for $A \rightarrow B$.

Open grounds: constructive functions.

E.g. $f(x, y, x^A)$ ground for $A(x) \vdash B(y)$ if, for every individuals p, q, for every g ground for A(p), f(p, q, g) ground for B(q).

<u>Non-primitive operations</u>: defined by equations for computing. E.g., for every $\rightarrow Ix^A(f(x^A))$ ground for $A \rightarrow B$, for every g ground for A,

$$\rightarrow E(\rightarrow Ix^A(f(x^A)),g) = f(g).$$

With valid arguments: valid arguments \implies valid inferences.

With grounds, performance of inference is application of an operation on grounds for the premises. An inference is valid if the applied operation produces grounds for the conclusion from grounds for the premises. A proof is a chain of valid inferences.

Prawitz's provability does not reduce to derivability in a recursive system because of Gödel's incompleteness. Nonetheless, any sound system of this kind exemplify valid arguments or grounds, and can be expanded to a more powerful one through semantic principles.

Prawitz's concept of proof may be understood as *inexhaustible relative to* given formal configurations. We have a kind of dialectic between the formal and the informal (semantic) level, reminding of Kreisel's dialectic between the formal and the informal.

First, Gödel does not think that the notion of proof should be defined in terms of valid inferences. He did not considered a proof as a

sequences of expressions satisfying certain formal conditions, but a sequence of thoughts convincing a sound mind. (Gödel 1995)

The approach is <u>non-local</u>. In spite of the reference to Turing's notion of computability, Gödel underlines that systematic methods

for actualising the development of our understanding of the abstract terms implied in a proof can, for humans, (contrary to computers), converge towards an infinity of distinguishable states of mind. (Crocco 2019)

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Second. Gödel seems to consider strict absolute proofs as untyped objects. Strict absoluteness does not simply means independence from any recursive system, but independence from any language.

Renounce "labelling" the components of a proof, apart from their being such. Renounce "typing" proofs. Gödel's approach could be conceived of as aiming to single out the "movements" of a proof-act.

Proof-nets are developed by Girard for the multiplicative fragment of Linear Logic (MLL), whose complete one-sided calculus is

$$\begin{array}{c} \hline & \vdash \Gamma, A & \vdash \Delta, A^{\perp} \\ \hline & \vdash \Gamma, A^{\perp} \end{array} \texttt{id} \quad \hline & \vdash \Gamma, \Delta & \vdash \Delta, A^{\perp} \\ \hline & \vdash \Gamma, A, B \\ \hline & \vdash \Gamma, A ~ \Im ~ B \end{array} \Im \quad \hline & \vdash \Gamma, A & \vdash \Delta, B \\ \hline & \vdash \Gamma, \Delta, A \otimes B \end{array} \otimes \end{array}$$

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The derivation on the left π_1 is different from the derivation on the left π_2 although they are "morally" equivalent

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A proof structure \mathfrak{S} is a directed graph with vertices labeled by MLL-formulas and edges labeled by elements of $\{id, \mathfrak{P}, \otimes, cut\}$ according to the following conditions



A switching σ of a MLL proof structure \mathfrak{S} is a function associating to each \mathfrak{P} -node in \mathfrak{S} a switch, i.e. a block of the partition $\langle [1], [2] \rangle$. For each switching, we define $\sigma(\mathfrak{B})$ as the undirected correction graph (also called test) obtained by forgetting the orientation of edges and by removing, for each \mathfrak{P} -node with conclusion $A \mathfrak{P} B$, the edge labeled by B if its switch is [1] or the edge labeled by A if the switch is [2].

A proof structure \mathfrak{S} is a *proof net* if, and only if each test $\sigma(\mathfrak{B})$ of \mathfrak{B} is acyclic and connected.



There is a function f from the set of the derivations in \mbox{MLL} to proof nets such that

- $\bullet \ f \text{ is surjective and}$
- **2** $f(\pi) = f(\rho)$ if, and only if, π and ρ are equal modulo permutation of rules

The two derivations below differ only in the order of application of the rules \otimes . They are mapped on the same proof-net.

$$\begin{array}{c|c} \hline & & \\ \hline \hline & & \\ \hline & & \\ \hline \hline & & \\ \hline \hline \\ \hline & & \\ \hline \hline & & \\ \hline \hline \\ \hline & & \\ \hline \hline \\ \hline \hline & & \\ \hline \hline \\ \hline \hline$$



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Prawitz & Gödel. Gödel ~ Girard. Hence, Prawitz & Girard?

Three differences between Prawitz and Girard:

- Prawitz first-order, Girard second-order;
- Prawitz formulas-as-types conception, Girard untyped;
- Prawitz verificationism, Girard bidirectionalism.

BUT

- a) An approach to grounds in Ludics has been put forward in a paper of ours titled *Game of grounds*
- b) Naibo and Takahashi studied the concept of harmony in the light of C-ludics

So, maybe Prawitz ~ Girard. And hence it could be possible Prawitz ~ Gödel - **although indirectly**?

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Thank you

PS: this research project is partially connected to

D. Catta & A. Piccolomini d'Aragona (2020), *Game of grounds*, in G. Oliveri, S. Boscolo & C. Ternullo (eds), *Objects, structures and logics, Boston studies in the philosophy and history of science*, Springer